

Poincaré polynomial of the moduli spaces of parabolic bundles

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Abstract

In this paper we use Weil conjectures (Deligne's theorem) to calculate the Betti numbers of the moduli spaces of semi-stable parabolic bundles on a curve. The quasi parabolic analogue of the Siegel formula, together with the method of Harder-Narasimhan filtration gives us a recursive formula for the Poincaré polynomials of the moduli. We solve the recursive formula by the method of Zagier, to give the Poincaré polynomial in a closed form. We also give explicit tables of Betti numbers in small rank, and genera.

1 Introduction

This paper uses the Riemann hypothesis of Weil (Deligne's theorem) to explicitly determine the Betti numbers of the moduli of semistable parabolic bundles on a curve (when parabolic semi-stability implies parabolic stability).

Vector bundles with parabolic structures were introduced by Seshadri, and their moduli was constructed by Mehta-Seshadri (see [S] for an account). Our approach to the calculation of the Betti numbers is an extension of the method used by Harder and Narasimhan [H-N] in the case of ordinary vector bundles. Harder and Narasimhan use the result of Siegel that the Tamagawa number of SL_r over a function field of transcendence degree one over a finite field is 1. This result can be reformulated purely in terms of vector bundles to

give the formula (equation (2.16)), which was used by Desale and Ramanan [D-R] in their refinement of the Harder-Narasimhan Betti number calculation.

In place of the above formula, we use its quasi-parabolic analogue (see equation (2.19)) proved by Nitsure [N2], to extend the calculation of Harder and Narasimhan, as refined by Desale and Ramanan, to parabolic case.

This gives us a recursive formula to obtain Betti numbers. Such a recursive formula had been obtained earlier for genus ≥ 2 by Nitsure[N1] using the Yang-Mills method of Atiyah-Bott[A-B], and this was extended to lower genus by Furuta and Steer[F-S].

Finally following Zagier's[Z] method of solving such a recursion(in the case of ordinary vector bundles), we obtain an explicit formula for the Poincaré polynomials. We give sample tables in lower ranks and genera.

This paper is arranged as follows. In section 2, we have introduced our notations and recalled certain basic facts about parabolic bundles for the convenience of the reader. The paper of Desale and Ramanan computes the Poincaré polynomial of the moduli space of stable bundles, starting with the formula of Siegel (2.16). In section 3, we have followed their general pattern with suitable changes needed to handle the parabolic case, with the Siegel formula replaced by its parabolic analogue (2.19). This gives us the theorem (3.36), which is our desired recursive formula for the Poincaré polynomial. Along the way, we need a certain substitution ($\omega_i \rightarrow -t^{-1}$, $q \rightarrow t^{-2}$) used by Harder and Narasimhan, who have sketched its justification. We give a detailed proof of why such a substitution works (in a somewhat more general context) in section 4. In section 5, we solve the recursive formula using Zagier[Z]'s approach, to get the explicit form (5.23) of the Poincaré polynomial. In section 6, we give some sample computations of the Poincaré polynomials of these moduli spaces and check their dependence on the weights and the degree when the rank is low (2,3 and 4). In the appendix (section 7), we have given tables for the Betti numbers of these moduli spaces in rank 2,3 and 4.

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2 Basic definitions and notations

Zeta function of a curve.

Let \mathbb{F}_q be a finite field, and let $\overline{\mathbb{F}}_q$ be its algebraic closure. Let X be a smooth projective geometrically irreducible curve over \mathbb{F}_q , where geometric irreducibility means $\overline{X} = X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$ is irreducible.

Given any integer $r > 0$, let $\mathbb{F}_{q^r} \subset \overline{\mathbb{F}}_q$ be the unique field extension of degree r over \mathbb{F}_q . Let $N_r = |X(\mathbb{F}_{q^r})|$ be the cardinality of the set of \mathbb{F}_{q^r} -rational points of X . Recall that the zeta function of X is defined by

$$Z_X(t) = \exp \left(\sum_{r>0} \frac{N_r t^r}{r} \right) \quad (2.1)$$

By the Weil conjectures it follows that the zeta function has the form

$$Z_X(t) = \frac{\prod_{i=1}^{2g} (1 - \omega_i t)}{(1-t)(1-qt)} \quad (2.2)$$

where ω_i 's are algebraic integers of norm $q^{1/2}$, and g is the genus of the curve. For $\nu \geq 1$, let X_ν denote the curve $X_\nu = X \otimes_{\mathbb{F}_q} \mathbb{F}_{q^\nu}$. The following remark will be used later.

Remark 2.3. If the zeta function of X over \mathbb{F}_q is as given in (2.2), then the zeta function for the curve X_ν over \mathbb{F}_{q^ν} has the form

$$Z_{X_\nu}(t) = \frac{\prod_{i=1}^{2g} (1 - \omega_i^\nu t)}{(1-t)(1-qt)}. \quad (2.4)$$

Rational points on Flag varieties

Now we recall the computation of the number of rational points of flag varieties. Let $k = \mathbb{F}_q$ as before, let n and m be positive integers, and let there be given non-negative integers r_1, \dots, r_m with $r_1 + \dots + r_m = n$. We denote by $\text{Flag}(n, m, (r_j))$ the variety of all flags $k^n = F_1 \supset \dots \supset F_m \supset F_{m+1} = 0$ of vector subspaces in k^n , with $\dim(F_j/F_{j+1}) = r_j$.

Proposition 2.5. *The number of \mathbb{F}_q -rational points of $\text{Flag}(n, m, (r_j))$ is*

$$f(q, n, m, (r_j)) = \frac{\prod_{i=1}^n (q^i - 1)}{\prod_{\{j|r_j \neq 0\}} \prod_{l=1}^{r_j} (q^l - 1)} \quad (2.6)$$

Proof. The number of rational points $g(r, p)$ on the Grassmanian $\text{Grass}(r, p)$ of p -dimensional subspaces of k^r can be seen to be

$$g(r, p) = \frac{(q^r - 1) \cdots (q^r - q^{p-1})}{(q^p - 1) \cdots (q^p - q^{p-1})} \quad (2.7)$$

and the number $f(q, n, m, (r_j))$ clearly satisfies

$$f(q, n, m, (r_j)) = g(n, r_m)g(n - r_m, r_{m-1}) \cdots g(r_1 + r_2, r_2) \quad (2.8)$$

Simplifying yields the desired formula. \square

Parabolic vector bundles

Let $S = \{P_1, \dots, P_s\}$ be any closed subset of X whose points are k -rational. For each $P \in S$, let there be given a positive integer m_P .

We fix an indexed family of real numbers (α_i^P) , where $P \in S$ and $i = 1, \dots, m_P$, satisfying $0 \leq \alpha_1^P < \alpha_2^P < \dots < \alpha_{m_P}^P < 1$, which we denote simply by α . We fix the set S , the integers (m_P) and the family α in all that follows.

The parabolic weights at P of the parabolic bundles that we will consider in this paper are going to belong to the chosen set $\{\alpha_1^P, \alpha_2^P, \dots, \alpha_{m_P}^P\}$. (Note that this property will be inherited by the sub-quotients of such parabolic bundles.) This allows us to formulate the definition of parabolic bundles in a somewhat different way from Seshadri, which is more suited for our inductive arguments. However, the difference is only superficial, and we explain later (remark (2.11) on the next page) the bijective correspondence between parabolic bundles in our sense, and parabolic bundles in Seshadri's sense which have weights in our given set.

A **quasi-parabolic data** R (or simply 'data' when the context is clear) is an indexed family of non-negative integers (R_i^P) for $P \in S$ and $1 \leq i \leq m_P$, satisfying the following condition: $\sum_{i=1}^{m_P} R_i^P$ is a positive integer independent of $P \in S$. We call $n(R) = \sum_{i=1}^{m_P} R_i^P$ as the **rank** of the quasi-parabolic data R .

Let L be another quasi-parabolic data. We say L is a **sub-data** of a given data R if $L_i^P \leq R_i^P$ for all P and i , and $n(L) < n(R)$. We also define its **complementary sub-data** $R - L$ by $(R - L)_i^P = R_i^P - L_i^P$.

A quasi-parabolic structure with data R on a vector bundle E on X , by definition, consists of a flag

$$E^P = E_1^P \supset E_2^P \dots \supset E_{m_P}^P \supset E_{m_P+1}^P = 0 \quad (2.9)$$

of vector subspaces in the fiber E^P over each point P of S , such that $R_i^P = \dim(E_i^P/E_{i+1}^P)$ for $P \in S$ and $1 \leq i \leq m_P$.

A parabolic structure with data R on a vector bundle E is a quasi-parabolic structure on E with data R along with weights α_i^P for each P and i . We say R_i^P is the multiplicity of the weight α_i^P . To a data R , we associate the real number $\alpha(R)$ by

$$\alpha(R) = \sum_P \sum_{i=1}^{m_P} R_i^P \alpha_i^P. \quad (2.10)$$

Remark 2.11. We record here the minor changes in notations and conventions that we have made (compared to the original notation of Seshadri). In our definition, note that the data R has the property that $R_i^P \geq 0$ (and not > 0), hence if E is a parabolic bundle in our sense with data R then the inclusions occurring in the filtration (2.9) are not necessarily strict. We recover the definition of Seshadri by re-defining the weights $\bar{\alpha}$ inductively as follows

$$\begin{aligned} \bar{\alpha}_1^P &= \min_i \{\alpha_i^P | R_i^P \neq 0\} \\ \bar{\alpha}_j^P &= \min_k \{\alpha_k^P | R_k^P \neq 0, \alpha_k^P - \bar{\alpha}_{j-1}^P > 0\}. \end{aligned} \quad (2.12)$$

From this it follows that each $\bar{\alpha}_j^P$ equals α_i^P for exactly one i , which allows us to define $\bar{R}_j^P = R_i^P$ for that particular i . Now it is clear that E is a parabolic bundle in the sense of Seshadri with weights $\bar{\alpha}$ and multiplicities \bar{R} , where the flags are defined by sub-spaces $\bar{E}_j^P = E_i^P$ for that i for which $\bar{\alpha}_j^P = \alpha_i^P$. Since we have fixed the weights α , we can recover the parabolic bundle in our sense from a given parabolic bundle with weights $\bar{\alpha}$ and multiplicities \bar{R} in the sense of Seshadri when the set $\{\bar{\alpha}_i^P\}$ is a subset of $\{\alpha_i^P\}$ for each P , by simply assigning

$$\begin{aligned} R_i^P &= 0 \text{ if } \alpha_i^P \neq \bar{\alpha}_j^P \text{ for any } j \\ &= \bar{R}_j^P \text{ if } \alpha_i^P = \bar{\alpha}_j^P \text{ for some } j \end{aligned} \quad (2.13)$$

and the defining the sub-spaces occurring in the flags inductively by

$$\begin{aligned} E_1^P &= E^P \\ E_i^P &= E_{i-1}^P \text{ if } \alpha_i^P \neq \bar{\alpha}_j^P \text{ for any } j \\ &= \bar{E}_j^P \text{ if } \alpha_i^P = \bar{\alpha}_j^P \text{ for some } j \end{aligned} \quad (2.14)$$

This sets up a bijective correspondence between parabolic bundles in our sense and in the sense of Seshadri. Also note that

$$\sum_P \sum_i R_i^P \alpha_i^P = \sum_P \sum_i \bar{R}_i^P \bar{\alpha}_i^P, \quad (2.15)$$

which will enable us to write the parabolic degree in terms of our modified definition. The advantage of our definition is that it is easier to handle the induced parabolic structures on the sub-bundles and the quotient bundles in what follows. Also the parabolic homomorphisms between two parabolic bundles E and E' with data R and R' respectively in our sense, having the same fixed family of weights (α_i^P) , are just filtration preserving homomorphisms of the vector bundles.

Quasi-parabolic Siegel formula

For a positive integer n and for any line bundle \mathcal{L} on X , let $J_n(\mathcal{L})$ denote the set of isomorphism classes of vector bundles E on X with $\text{rank}(E) = n$ and determinant \mathcal{L} . Let $|\text{Aut}(E)|$ denote the cardinality of the group of all automorphisms of E . Then the Siegel formula, asserts that

$$\sum_{E \in J_n(\mathcal{L})} \frac{1}{|\text{Aut}(E)|} = \frac{q^{(n^2-1)(g-1)}}{q-1} Z_X(q^{-2}) \cdots Z_X(q^{-n}) \quad (2.16)$$

The above formula was given a proof purely in terms of vector bundles by Ghione and Letizia [G-L].

For a line bundle \mathcal{L} on X , let $J_R(\mathcal{L})$ denote the set of all isomorphism classes of quasi-parabolic vector bundles with data R , and determinant \mathcal{L} . Let $f_R(q)$ (denoted by $f(q, R)$ in [N2]) be the number of \mathbb{F}_q -valued points of the variety $\mathcal{F}_R = \prod_{P \in S} \text{Flag}(n(R), m_P, (R_i^P))$ where $\text{Flag}(n(R), m_P, (R_i^P))$ is the flag variety determined by (R_i^P) . Now by equation (2.6), we have

$$f_R(q) = \frac{\prod_{i=1}^{n(R)} (q^i - 1)^{|S|}}{\prod_{P \in S} \prod_{\{i | R_i^P \neq 0\}} \prod_l^{R_i^P} (q^l - 1)}. \quad (2.17)$$

Let $|\text{ParAut}(E)|$ denote the cardinality of the set of quasi-parabolic isomorphisms of a quasi-parabolic bundle E . The Siegel formula has the following quasi-parabolic analogue, which was proved by Nitsure [N2].

Theorem 2.18. (*Quasi-parabolic Siegel formula*)

$$\sum_{E \in J_R(\mathcal{L})} \frac{1}{|\text{ParAut}(E)|} = f_R(q) \frac{q^{(n(R)^2-1)(g-1)}}{q-1} Z_X(q^{-2}) \dots Z_X(q^{-n(R)}) \quad (2.19)$$

For example, if S is empty or more generally if the quasi-parabolic structure at each point of S is trivial (that is, each flag consists only of the zero subspace and the whole space), then on one hand $\text{ParAut}(E) = \text{Aut}(E)$, and on the other hand each flag variety is a point, and so $f_R(q) = 1$. Hence in this situation the above formula reduces to the original Siegel formula.

Parabolic degree and stability

Let E be a parabolic bundle over X with data R . Because of (2.15), we can define the parabolic degree of E and the parabolic slope of E as follows:

$$\text{pardeg}(E) = \deg(E) + \alpha(R) \quad \text{and} \quad \text{par}\mu(E) = \text{pardeg}(E)/\text{rank}(E). \quad (2.20)$$

A parabolic bundle E on X is said to be parabolic stable (resp. parabolic semi-stable) if for every non-trivial proper sub-bundle F of E with induced parabolic structure, we have $\text{par}\mu(F) < \text{par}\mu(E)$ (resp. \leq).

The equation (2.15) implies that the definitions of parabolic stable (resp. parabolic semi-stable) bundles are not altered by the change in the definition of the parabolic bundles we have made.

We say that the numerical data (d, R) satisfies the condition ‘par semi-stable = par stable’ if every parabolic semistable bundle with data R and degree d is automatically parabolic stable.

Remark 2.21. If the degree d and rank $n(R)$ are coprime and all the weights are assumed to be very small ($\alpha_i^P < 1/(n(R)^2|S|)$ for example) then, each parabolic semistable bundle is actually parabolic stable.

We now recall the following.

Lemma 2.22. *If E is a parabolic stable bundle, then every parabolic homomorphism of E into itself is a scalar endomorphism.*

Parabolic Harder-Narasimhan-Intersection types

Recall the following.

Proposition 2.23. *Any non-zero parabolic bundle E with the data R admits a unique filtration by sub-bundles*

$$0 = G_0 \subsetneq G_1 \subsetneq \dots \subsetneq G_r = E \quad (2.24)$$

satisfying

- i) G_i/G_{i-1} is parabolic semi-stable for $i = 1, \dots, r$
- ii) $\text{par}\mu(G_i/G_{i-1}) > \text{par}\mu(G_{i+1}/G_i)$ for $i = 1, \dots, r-1$.

Equivalently,

- 1) G_i/G_{i-1} is parabolic semi-stable for $i = 1, \dots, r$
- 2) For any parabolic sub-bundle F of E containing G_{i-1} we have $\text{par}\mu(G_i/G_{i-1}) > \text{par}\mu(F/G_{i-1})$, $i = 1, \dots, r$.

The result is first proved over an algebraically closed field, and then Galois descent is applied (using the uniqueness of the filtration) to prove that the filtration is defined over the original field.

The unique filtration is called the parabolic Harder-Narasimhan filtration.

Let $(I_{i,k}^P)$ be an indexed collection of non-negative integers, where $P \in S$, $1 \leq i \leq m_P$, and $1 \leq k \leq r$ where r is a given positive integer. We say that $I = (I_{i,k}^P)$ is a **partition** of R of length r if the following holds:

- 1) For $P \in S$ and $1 \leq i \leq m_P$, we have $\sum_{k=1}^r I_{i,k}^P = R_i^P$.
- 2) For $P \in S$ and $1 \leq k \leq r$, the summation $\sum_{i=1}^{m_P} I_{i,k}^P$ is independent of P .¹
- 3) Given any $k \leq r$, $I_{i,k}^P \neq 0$ for some P and i .

We write $\ell(I) = r$ to indicate that I has length r .

Suppose I is a partition of R with $\ell(I) = r$. For $j = 1, \dots, r$ define a sub-data R_j^I of R by the equality $(R_j^I)_i^P = I_{i,j}^P$. We also define the sub-data $R_{\leq j}^I$ (resp. $R_{\geq j}^I$) of R by the equality

$$(R_{\leq j}^I)_i^P = \sum_{k \leq j} I_{i,k}^P \quad (\text{resp.} \quad (R_{\geq j}^I)_i^P = \sum_{k \geq j} I_{i,k}^P). \quad (2.25)$$

¹For later reference, this number will be equal to the rank of G_k .

Note that the rank $n(R_{\leq j}^I)$ of $R_{\leq j}^I$ is equal to $n(R_1^I) + n(R_2^I) \dots + n(R_j^I)$. We observe that the partition I of R induces a partition $I_{\leq j}$ (resp. $I_{\geq j}$) on $R_{\leq j}^I$ (resp. $R_{\geq j}^I$) defined by $(I_{\leq j})_{i,k}^P = I_{i,k}^P$ (resp. $(I_{\geq j})_{i,k}^P = I_{i,k}^P$), where $k \leq j$ (resp. $k \geq j$).

We now recall how partitions, as abstractly defined above, are associated with parabolic bundles in Nitsure[N1]. To each $E \in J_R(\mathcal{L})$ we have the parabolic Harder-Narasimhan filtration $0 \subset G_1 \subset \dots \subset G_r = E$ which gives a filtration on the fibers. Then the **intersection matrix** $(I_{i,k}^P)$ corresponding to it is defined in [N1], by putting

$$I_{m_P,1}^P = \dim(E_{m_P}^P \cap G_1^P) \quad (2.26)$$

and

$$I_{j,l}^P = \dim(E_j^P \cap G_l^P) - \sum_{\substack{i \leq j, \text{ and } k \leq l \\ (i,k) \neq (l,j)}} I_{i,k}^P \quad (2.27)$$

with this definition I becomes a partition of R with $\ell(I) = r$. The sub-bundles G_j (resp. quotients E/G_j) under the induced parabolic structure have the sub-data $R_{\leq j}^I$ (resp. $R_{\geq j+1}^I$). Also the sub-quotient G_j/G_{j-1} has the sub-data R_j^I .

Moduli spaces

For the moment assume that our ground field k is algebraically closed. Recall that parabolic semistable bundles, with a fixed parabolic slope, form an abelian category with the property that each object has finite length and simple objects are precisely the parabolic stable bundles. Hence for every parabolic semi-stable bundle E there exists a Jordan-Holder series

$$E = E_r \supset E_{r-1} \dots \supset E_1 \supset 0$$

such that E_i/E_{i-1} is a parabolic stable bundle satisfying $\text{par}\mu(E_i/E_{i-1}) = \text{par}\mu(E)$. If we write $Gr(E)$ for $\oplus_i E_i/E_{i-1}$, then it is well defined and is a parabolic semistable bundle with the same data as E . We say that two parabolic semistable bundles E and F are S-equivalent if $Gr(E)$ and $Gr(F)$ are isomorphic as parabolic bundles.

Mehta and Seshadri[M-S] prove that there exists a coarse moduli scheme $\mathcal{M}_{R,\mathcal{L}}$ of the S-equivalence classes of parabolic semistable bundles with the

data R and determinant \mathcal{L} . The scheme $\mathcal{M}_{R,\mathcal{L}}$ is a normal projective variety. Further the subset $\mathcal{M}_{R,\mathcal{L}}^s$ of $\mathcal{M}_{R,\mathcal{L}}$ corresponding to parabolic stable bundles is a smooth open subvariety.

Remark 2.28. Our method computes the Betti numbers of the moduli space of parabolic bundles for any curve over \mathbb{C} because of the following reason. The theorem of Seshadri implies that the topological type of these moduli spaces depend only on the genus g , the cardinality of parabolic vertices $|S|$, the degree d and the set of weights α along with their multiplicities R . We start with such a data, construct a smooth projective absolutely irreducible curve X over a finite field $k = \mathbb{F}_q$ which has at least $|S|$ number of k -rational points (by taking $q = p^n$ for large n , or $q = p$ for a large prime p). The use of Witt vectors allows us to spread the curve and the moduli spaces to the quotient field of the ring of Witt vectors, when the condition ‘par semi-stable = par stable’ holds. Now by Weil conjectures it follows that the Betti numbers of the moduli space of parabolic bundles over X coincides with the one over the curve obtained by the change of base to \mathbb{C} .

Parabolic extensions

Let E', E and E'' be parabolic bundles with data R', R and R'' respectively. Let

$$0 \longrightarrow E' \xrightarrow{i} E \xrightarrow{j} E'' \longrightarrow 0 \quad (2.29)$$

be a short exact sequence of the underlying vector bundles such that the parabolic structures induced on E' and E'' from the given parabolic structure on E coincide with the given parabolic structures of E' and E'' , we say that (2.29) is a short exact sequence of parabolic bundles. We also say $[E] = (E, i, j)$ is a **parabolic extension** of E'' by E' .

We say two parabolic extensions $[E_1]$ and $[E_2]$ are equivalent if there exists an isomorphism of parabolic bundles $\gamma : E_1 \longrightarrow E_2$ such that the following diagram with commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E' & \xrightarrow{i_1} & E_1 & \xrightarrow{j_1} & E'' & \longrightarrow & 0 \\ & & \parallel & & \downarrow \gamma & & \parallel & & \\ 0 & \longrightarrow & E' & \xrightarrow{i_2} & E_2 & \xrightarrow{j_2} & E'' & \longrightarrow & 0 \end{array} \quad (2.30)$$

We denote the set of equivalence classes of parabolic extensions by $\text{ParExt}(E'', E')$. The proof of the following lemma is straight-forward and we omit it.

Lemma 2.31. *There is a canonical bijection between $\text{ParExt}(E'', E')$ and $H^1(X, \mathcal{ParHom}(E'', E'))$, where $\mathcal{ParHom}(E'', E')$ is the sheaf of germs of parabolic homomorphisms from E'' to E' .*

By analogy with the case of ordinary vector bundles, we define an action of $\text{ParAut}(E'') \times \text{ParAut}(E')$ on $\text{ParExt}(E'', E')$ as follows: Given automorphisms $\alpha \in \text{ParAut}(E'')$, $\beta \in \text{ParAut}(E')$ and a parabolic extension $[E] = (E, i, j) \in \text{ParExt}(E'', E')$ we define the parabolic extension $\beta[E]\alpha$ to be the extension $(E, \beta \circ i, j \circ \alpha)$.

Now fix a parabolic extension $[E]$ of E'' by E' . The proof of the following lemma is analogous to the corresponding statement for ordinary vector bundles.

Lemma 2.32. (a) *The orbit of $[E]$ under this action is the set of equivalence class of parabolic extensions which have their middle terms isomorphic to E as parabolic bundles.*

(b) *The stablizer of $[E]$ under this action is precisely the subgroup of $\text{ParAut}(E'') \times \text{ParAut}(E')$ consisting of elements of the form (α, β) such that there exists a parabolic automorphism of E which takes E' to itself and induces α on E'' and β on E' .*

For the convenience of the reader we summerize below, in one place the notations used in this paper. Some of the notations are introduced above while the rest will be introduced subsequently.

Summary of notation

X	= a smooth projective geometrically irreducible curve over the finite field \mathbb{F}_q .
\overline{X}	= the curve $X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$, where $\overline{\mathbb{F}}_q$ is an algebraic closure of \mathbb{F}_q .
$Z_X(t)$	= the zeta function of the curve X .
X_ν	= $X \otimes_{\mathbb{F}_q} \mathbb{F}_{q^\nu}$, where $\mathbb{F}_{q^\nu} \subset \overline{\mathbb{F}}_q$ is a finite field with q^ν elements.

For positive integers n and m and non-negative integers r_1, \dots, r_m with $r_1 + \dots + r_m = n$,

$\text{Flag}(n, m, r_j)$ = the variety of all flags $k^n = F_1 \supset \dots \supset$

	$F_m \supset F_{m+1} = 0$ of vector subspaces in k^n , with $\dim(F_j/F_{j+1}) = r_j$.
$ J(\mathbb{F}_q) $	= the number of \mathbb{F}_q -rational points of the Jacobian of X .
S	= a finite set of k -rational points of X . (These are the parabolic vertices.)
m_P	= a fixed positive integer defined for each $P \in S$. For $P \in S$, and $1 \leq i \leq m_P$,
α	= (α_i^P) is the set of allowed weights. For $P \in S$, and $1 \leq i \leq m_P$,
R	= (R_i^P) , the quasi-parabolic data (or simply ‘data’).
$n(R)$	= $\sum_{i=1}^{m_P} R_i^P$, the rank of the data R .
L	= a sub-data of R and
$R - L$	= the complementary sub-data defined by $(R - L)_i^P = R_i^P - L_i^P$.
\mathcal{L}	= a line bundle on X .
E	= a vector bundle with a parabolic structure with data R .
$J_R(\mathcal{L})$	= the set of isomorphism classes of quasi-parabolic vector bundles with data R , and determinant \mathcal{L} .
$\alpha(R)$	= $\sum_P \sum_{i=1}^{m_P} R_i^P \alpha_i^P$, the parabolic contribution to the degree.
$\deg(E)$	= the ordinary degree of E .
$\text{pardeg}(E)$	= $\deg(E) + \alpha(R)$, the parabolic degree of E .
$\text{par}\mu(E)$	= $\text{pardeg}(E)/\text{rank}(E)$, the parabolic slope of E . For $P \in S$, $1 \leq i \leq m_P$, and $1 \leq k \leq r$,
I	= $(I_{i,k}^P)$, the intersection type of Nitsure, which is a partition of R .
$\ell(I)$	= the length of the intersection type I . For $j \leq r$, we have
R_j^I	= the sub-data defined by $(R_j^I)_i^P = I_{i,j}^P$,
$R_{\leq j}^I$	= the sub-data defined by $(R_{\leq j}^I)_i^P = \sum_{k \leq j} I_{i,k}^P$,

$$\begin{aligned}
R_{\geq j}^I &= \text{the sub-data defined by } (R_{\geq j}^I)_i^P = \sum_{k \geq j} I_{i,k}^P. \\
I_{\leq j} &= \text{the partition of } R_{\leq j}^I \text{ defined by } (I_{\leq j})_{i,k}^P = I_{i,k}^P \\
&\quad \text{where } k \leq j. \\
I_{\geq j} &= \text{the partition of } R_{\geq j}^I \text{ defined by } (I_{\geq j})_{i,k}^P = I_{i,k}^P \\
&\quad \text{where } k \geq j. \\
\mathcal{M}_{R,\mathcal{L}} &= \text{the moduli space of parabolic semistable bundles} \\
&\quad \text{with the data } R \text{ and determinant } \mathcal{L}. \\
\mathcal{M}_{R,\mathcal{L}}^s &= \text{the open sub variety of } \mathcal{M}_{R,\mathcal{L}} \\
&\quad \text{corresponding to the parabolic stable bundles.}
\end{aligned}$$

$$\begin{aligned}
&\text{For parabolic bundles } E', E \text{ and } E'' \text{ with} \\
&\text{data } R', R \text{ and } R'', \text{ we denote by} \\
[E] &= (E, i, j), \text{ a parabolic extension of } E'' \text{ by } E'. \\
\text{ParExt}(E'', E') &= \text{the set of equivalence classes of parabolic} \\
&\quad \text{extensions of } E'' \text{ by } E'. \\
\beta_R(\mathcal{L}) &= \sum (1/|\text{ParAut}(E)|) \\
&\quad \text{where summation is over all } E \in J_R(\mathcal{L}) \text{ such} \\
&\quad \text{that } E \text{ is parabolic semistable.} \\
J_R(\mathcal{L}, I) &= \text{the set of isomorphism classes of parabolic} \\
&\quad \text{bundles with weights } \alpha, \text{ of intersection} \\
&\quad \text{type } I, \text{ and determinant } \mathcal{L}. \\
\beta_R(\mathcal{L}, I) &= \sum (1/|\text{ParAut}(E)|), \\
&\quad \text{where the summation is over all } E \text{ in} \\
&\quad J_R(\mathcal{L}, I).
\end{aligned}$$

$$\begin{aligned}
&\text{We also write} \\
\beta_R(d, I) &= \beta_R(\mathcal{L}, I), \text{ since } \beta_R(\mathcal{L}, I) \\
&\quad \text{depends on } \mathcal{L} \text{ only via its degree } d = \deg(\mathcal{L}). \\
\mathcal{F}_R &= \prod_{P \in S} \text{Flag}(n(R), m_P, R_i^P) \\
f_R(q) &= \text{the number of } \mathbb{F}_q\text{-valued points of the variety } \mathcal{F}_R. \\
C(I; d_1, \dots, d_r) &= \text{the integer defined by equation (3.8).} \\
\sigma_k(I) &= \sum_{P \in S} \sum_{i > t} \sum_{l < r-k+1} I_{i,r-k+1}^P I_{t,l}^P. \\
\sigma_R(I) &= \sum_k \sigma_k(I).
\end{aligned}$$

$\chi(F)$	For a vector bundle F
$\chi\left(\begin{smallmatrix} \nu_1 & \dots & \nu_r \\ \delta_1 & \dots & \delta_r \end{smallmatrix}\right)$	= the Euler characteristic. = the numerical function of Desale-Ramanan defined by the equation (3.20).
\sum_{\circ}	denotes the summation over all $(d_1, \dots, d_r) \in \mathbb{Z}^r$ with $\sum_i d_i = d$ and satisfying equation (3.7).
$\tau_{n(R)}(q)$	$= \frac{q^{(n(R)^2-1)(g-1)}}{q-1} Z_X(q^{-2}) \dots Z_X(q^{-n(R)}).$
$\tilde{f}_R(t)$	= the rational function corresponding to f_R given by the equation (3.30).
$\tilde{\tau}_{n(R)}(t)$	= the rational function corresponding to $\tau_{n(R)}$ given by the equation (3.31).
$Q_{R,d}(t)$	$= t^{n(R)^2(g-1)}(1+t^{-1})^{2g} \tilde{\beta}_R(d),$ this is the main function for the recursion.
$Q_R(t)$	$= t^{n(R)^2(g-1)} \tilde{f}_R(t) \tilde{\tau}_{n(R)}(t).$
$P_{R,d}$	= the power series whose coefficients compute the Betti numbers of the moduli space of parabolic stable bundles with data R and degree d .
$N_R(I; d_1, \dots, d_r)$	= the integer given by the formula (3.38).
Y	= a smooth projective variety over \mathbb{F}_q .
N_ν	= the number of \mathbb{F}_{q^ν} -rational points of Y .
ω_i	For $i = 1, \dots, 2g$, we have = a fixed algebraic integer of norm $q^{1/2}$.
$h(u, v_1, \dots, v_{2g})$	= a rational function given by the equation (4.2)
$p(u, v_1, \dots, v_{2g})$	= the numerator occuring in the equation (4.2).
$(a_{J,j})$	= the coefficients occuring in (4.3).
J	= the multi-index $J = (i_1, i_2, \dots, i_{2g})$,
$ J $	$= \sum_{r=1}^{2g} i_r$, and
v^J	$= v_1^{i_1} v_2^{i_2} \dots v_{2g}^{i_{2g}}.$
N	= the ‘weighted degree’ of $p(u, v_1, v_2, \dots, v_{2g})$.
$b_{J,j}$	= the coefficients of h defined in (4.8).
$f_{\geq 0}(u, v_1, \dots, v_{2g})$	= the function defined by the equation (4.13).
M_r	$= f_{\geq 0}(q^r, \omega_1^r, \dots, \omega_{2g}^r).$

$$\begin{aligned}
Z_1(t) &= \text{the formal power series defined in (4.14).} \\
Z_2(t) &= \text{the formal power series defined in (4.15).} \\
Z(t) &= Z_1(t)Z_2(t).
\end{aligned}$$

For a meromorphic function h on a disc in \mathbb{C} , and $\alpha > 0$,

$$\begin{aligned}
\mu(h, \alpha) &= \text{the number of zeros minus the number of poles counted with multiplicities of } h \text{ with norm } \alpha. \\
P(T) &= \text{the polynomial defined by the equation (4.19).} \\
M'_R(I; d) &= \text{the integer given by the formula (5.4).}
\end{aligned}$$

For a real number λ ,

$$\begin{aligned}
M_R(I; \lambda) &= \text{the integer given by the formula (5.5).} \\
Q_{R,d}^\lambda(t) &= \text{the rational function defined in (5.7).} \\
S_{R,d}^\lambda(t) &= \text{the rational function defined in (5.8).} \\
\sum_{\circ_\lambda} &\text{denotes the summation over } (d_1, \dots, d_r) \in \mathbb{Z}^r
\end{aligned}$$

such that $\sum_i d_i = d$ and the equation (5.9) holds.

$$\begin{aligned}
Q_{R,d}^{\lambda-} &= Q_{R,d}^{\lambda-\epsilon} \text{ for } \epsilon \text{ small enough} \\
&\text{such that the function } Q_{R,d}^\lambda \text{ has no jumps in the interval } [\lambda - \epsilon, \lambda). \\
S_{R,d}^{\lambda-} &= S_{R,d}^{\lambda-\epsilon} \text{ for } \epsilon \text{ small enough} \\
&\text{such that the function } S_{R,d}^\lambda \text{ has no jumps in the interval } [\lambda - \epsilon, \lambda).
\end{aligned}$$

$$\begin{aligned}
\Delta Q_{R,d}^\lambda &= Q_{R,d}^\lambda - Q_{R,d}^{\lambda-} \\
\Delta S_{R,d}^\lambda &= S_{R,d}^\lambda - S_{R,d}^{\lambda-} \\
\delta_R(L) &= \text{the integer given by the equation (5.10).} \\
d(\lambda, L) &= n(L)\lambda - \alpha(L).
\end{aligned}$$

$$g_R(I; d) = \text{the rational function given by (5.17).}$$

$$\sigma'_R(I) = \sum_{P \in S} \sum_{k > l, i < t} I_{i,k}^P I_{t,l}^P.$$

$$M_g(I; \lambda) = \text{the integer given by the formula (5.25).}$$

$$P_R(t) = \text{the rational function (polynomial) defined by the equation (5.26).}$$

For a data R with rank $n(R) = 2$,

$$T = \text{the subset of } S \text{ consisting of parabolic vertices where the parabolic filtration is non-trivial.}$$

$$T_I = \{P \in T \mid I_{1,1}^P = 0\}.$$

$$\begin{aligned}
\chi_I &= \text{a characteristic function on } T, \text{ defined by} \\
&\quad \text{the equation (6.3).} \\
\psi_I &= \sum_{P \in T} \chi_I(P) (\alpha_1^P - \alpha_2^P). \\
a_I &= 1 \text{ if } d + [\psi_I] \text{ is even, and} \\
&= 0 \text{ if } d + [\psi_I] \text{ is odd.} \\
\delta^P &= \alpha_1^P - \alpha_2^P.
\end{aligned}$$

3 The inductive formula

The use of Parabolic-Harder-Narasimhan Intersection types

In this section we use the quasi-parabolic Siegel formula to obtain a recursive formula for the Poincaré polynomial of the moduli space of parabolic stable bundles when the condition ‘par semi-stable = par stable’ holds.

The left hand side of the quasi-parabolic Siegel formula (2.19) can be split into the summations coming from the parabolic semistable bundles and the unstable ones. In view of this we first define

$$\beta_R(\mathcal{L}) = \sum \frac{1}{|\text{ParAut}(E)|} \quad (3.1)$$

where summation is over all $E \in J_R(\mathcal{L})$ such that E is parabolic semistable.

We assume that the data R and degree d are so chosen that the condition ‘par semi-stable = par stable’ holds. In particular by lemma (2.22) this implies that for any such parabolic semistable bundle E , $|\text{ParAut}(E)| = q - 1$. Hence

$$|\mathcal{M}_{R,\mathcal{L}}(\mathbb{F}_q)| = (q - 1)\beta_R(\mathcal{L}) \quad (3.2)$$

is the number of \mathbb{F}_q -rational points of the moduli space of parabolic semistable bundles with the data R and determinant \mathcal{L} .

Now we have to take care of the unstable part of the summation (2.19). This summation can be further split into parabolic Harder Narasimhan intersection types. For these considerations, we make the following definitions:

Let I be a partition of R with $\ell(I) = r$. Let $J_R(\mathcal{L}, I)$ denote the set of isomorphism classes of parabolic bundles with data R , of intersection type I , and determinant \mathcal{L} .

Let

$$\beta_R(\mathcal{L}, I) = \sum \frac{1}{|\text{ParAut}(E)|} \quad (3.3)$$

where the summation is over all E in $J_R(\mathcal{L}, I)$. Note that $\beta_R(\mathcal{L}, I) = \beta_R(\mathcal{L})$ for the unique I which has $\ell(I) = 1$.

The summations occurring in (3.1) and (3.3) are finite because the parabolic bundles of fixed intersection type form a bounded family, so it is dominated by a variety, hence has only finitely many \mathbb{F}_q -rational points.

Now the quasi-parabolic Siegel formula (2.19) can be restated as

$$\sum_{r \geq 1} \sum_{\{I | \ell(I) = r\}} \beta_R(\mathcal{L}, I) = \frac{f_R(q) q^{(n(R)^2 - 1)(g-1)}}{q-1} Z_X(q^{-2}) \dots Z_X(q^{-n(R)}) \quad (3.4)$$

where $f_R(q)$ is given by (2.17).

Computation of the function $\beta_R(\mathcal{L}, I)$

The main step in the induction formula is to use the parabolic Harder - Narasimhan filtration to give a formula for $\beta_R(\mathcal{L}, I)$ when $\ell(I) > 1$, in terms of $\beta_{R'}(\mathcal{L}')$ of lower rank bundles. This we do in the following proposition which is an analogue of proposition (1.7) of Desale and Ramanan[D-R].

Proposition 3.5. (a) *The numbers $\beta_R(\mathcal{L}, I)$ and $\beta_R(\mathcal{L})$ depend on \mathcal{L} only via its degree $d = \deg(\mathcal{L})$ (hence they can be written as $\beta_R(d, I)$ and $\beta_R(d)$ resp.).*

(b) *$\beta_R(d, I)$ satisfies the following recursive relation*

$$\beta_R(d, I) = \sum_{\circ} q^{C(I; d_1, \dots, d_r)} |J(\mathbb{F}_q)|^{r-1} \prod_{k=1}^r \beta_{R_k^I}(d_k) \quad (3.6)$$

where \sum_{\circ} denotes the summation over all $(d_1, \dots, d_r) \in \mathbb{Z}^r$ with $\sum_i d_i = d$ and satisfying the following inequalities

$$\frac{d_1 + \alpha(R_1^I)}{n(R_1^I)} > \frac{d_2 + \alpha(R_2^I)}{n(R_2^I)} > \dots > \frac{d_r + \alpha(R_r^I)}{n(R_r^I)} \quad (3.7)$$

Here $|J(\mathbb{F}_q)|$ denotes the number of \mathbb{F}_q -valued points of the Jacobian of X , and

$$\begin{aligned} C(I; d_1, \dots, d_r) &= \sum_{P \in S} \sum_{k > l, i > t} I_{i,k}^P I_{t,l}^P - \sum_{k > l} (d_l n(R_k^I) - d_k n(R_l^I)) \\ &\quad + \sum_{k > l} n(R_l^I) n(R_k^I) (g-1) \end{aligned} \quad (3.8)$$

Proof. We prove both parts ((a) and (b)) of the proposition simultaneously by induction on $n = n(R)$. If $\ell(I) = 1$ then there is nothing to prove.

Consider a parabolic bundle E with data R , admitting the parabolic Harder-Narasimhan filtration $0 \subset G_1 \subset \dots \subset G_r = E$ of length $r \geq 2$. Let M be the quotient E/G_1 . If we give the induced parabolic structure to M then it has the data $R_{\geq 2}^I$.

Let T be the set of equivalence classes of parabolic extensions of M by G_1 which has the property that the middle term is isomorphic to E as a parabolic bundle. By lemma (2.32 (a)) T is same as the orbit of $[E]$ under the action of $\text{ParAut}(M) \times \text{ParAut}(G_1)$ on $\text{ParExt}(M, G_1)$, hence

$$|T| = \frac{|\text{ParAut}(M)| |\text{ParAut}(G_1)|}{|\text{stabilizer of } [E]|}. \quad (3.9)$$

Note that every parabolic automorphism of E takes G_1 to itself (hence also M). This implies that we get a group homomorphism

$$\text{ParAut}(E) \xrightarrow{\phi} \text{ParAut}(G_1) \times \text{ParAut}(M). \quad (3.10)$$

Now by lemma (2.32(b)) the stabilizer of $[E]$ is the image of ϕ , while the kernel of ϕ is equal to $I + H^0(X, \mathcal{P}\text{ar}\mathcal{H}\text{om}(M, G_1))$.

Combining all this we get

$$|T| = \frac{|\text{ParAut}(M)| |\text{ParAut}(G_1)| |\text{ParHom}(M, G_1)|}{|\text{ParAut}(E)|}. \quad (3.11)$$

By definition

$$\beta_R(\mathcal{L}, I) = \sum_{E \in J_R(\mathcal{L}, I)} \frac{1}{|\text{ParAut}(E)|} \quad (3.12)$$

which is

$$\sum_{(M, G_1)} \sum_{\mathcal{E}} \frac{1}{|\text{ParAut}(E)| |T|} \quad (3.13)$$

where the first summation extends over all pairs (M, G_1) with G_1 , a parabolic semistable bundle with data R_1^I , and M , parabolic bundle with data $R_{\geq 2}^I$ and intersection type $I_{\geq 2}$, such that $\det(M) \otimes \det(G_1) = \det(E)$. The second

summation extends over the set $\mathcal{E} = \text{ParExt}(M, G_1)$. By (3.11), the right hand side of the above expression (3.13) reduces to

$$\sum_{(M, G_1)} \frac{1}{|\text{ParAut}(M)| |\text{ParAut}(G_1)| q^{\chi(\mathcal{ParHom}(M, G_1))}}, \quad (3.14)$$

where

$$\chi(\mathcal{ParHom}(M, G_1)) = \dim_{\mathbb{F}_q}(\text{ParHom}(M, G_1)) - \dim_{\mathbb{F}_q}(\text{ParExt}(M, G_1)) \quad (3.15)$$

is the Euler characteristic of the sheaf $\mathcal{ParHom}(M, G_1)$.

We define certain numerical functions which depends only on the partition I as follows:

$$\sigma_k(I) = \sum_{P \in S} \sum_{i > t} \sum_{l < r-k+1} I_{i, r-k+1}^P I_{t, l}^P \quad \text{and} \quad \sigma_R(I) = \sum_k \sigma_k(I) \quad (3.16)$$

Then it can be checked that $\sigma_1(I)$ is the length of the torsion sheaf $\mathcal{S}_1(I)$, which is defined by the following exact sequence:

$$0 \longrightarrow \mathcal{ParHom}(M, G_1) \longrightarrow \mathcal{Hom}(M, G_1) \longrightarrow \mathcal{S}_1(I) \longrightarrow 0. \quad (3.17)$$

Using the fact that

$$\chi(\mathcal{ParHom}(M, G_1)) = \chi(M^* \otimes G_1) - \sigma_1(I), \quad (3.18)$$

the sum (3.14) becomes

$$= \sum_{M, G_1} \frac{q^{\sigma_1(I)}}{|\text{ParAut}(M)| |\text{ParAut}(G_1)| q^{\chi(M^* \otimes G_1)}}. \quad (3.19)$$

Recall that Desale-Ramanan[D-R] introduced certain numerical functions

$$\chi \begin{pmatrix} \nu_1 & \dots & \nu_r \\ \delta_1 & \dots & \delta_r \end{pmatrix} = \sum_{k > l} (\delta_l \nu_k - \delta_k \nu_l) + \sum_{k > l} \nu_l \nu_k (g-1). \quad (3.20)$$

With this definition of χ , we have the following equality

$$\chi \begin{pmatrix} n(R_1^I) & n(R) - n(R_1^I) \\ d_1 & d - d_1 \end{pmatrix} = \chi(M^* \otimes G_1) \quad (3.21)$$

as in [D-R]. Now by (3.21), the sum (3.19) equals

$$\sum_{d_1} q^{\sigma_1(I)-\chi} \binom{n(R_1^I) \quad n(R) - n(R_1^I)}{d_1 \quad d - d_1} \sum_{(\eta, \gamma)} \sum_M \frac{1}{\text{ParAut}(M)} \sum_{G_1} \frac{1}{\text{ParAut}(G_1)}, \quad (3.22)$$

where the first summation in (3.22) is over all integers d_1 with

$$(d_1 + \alpha(R_1^I))/n(R_1^I) > (d - d_1 + \alpha(R_{\geq 2}^I))/(n - n(R_1^I)). \quad (3.23)$$

The second summation in (3.22) is over isomorphism classes of line bundles η and γ such that $\eta \otimes \gamma = \mathcal{L}$. The third one is over all parabolic bundles M with data R , having intersection type $I_{\geq 2}$, and determinant η . The fourth summation in (3.22) is over all semi-stable parabolic bundles G_1 with data R_1^I , and determinant γ . This expression is equal to

$$\sum_q q^{\sigma_1(I)-\chi} \binom{n(R_1^I) \quad n - n(R_1^I)}{d_1 \quad d - d_1} \sum \beta_{R_{\geq 2}^I}(\eta, I_{\geq 2}) \beta_{R_1^I}(\gamma). \quad (3.24)$$

Now note that by induction, the terms inside the summation are independent of \mathcal{L} , hence part (a) of the proposition follows. From now on we write $\beta_R(d)$ and $\beta_R(d, I)$ for $\beta_R(\mathcal{L})$ and $\beta_R(\mathcal{L}, I)$.

By Desale-Ramanan [D-R] we have the relation

$$\chi \binom{n(R_1^I) \quad n - n(R_1^I)}{d_1 \quad d - d_1} + \chi \binom{n(R_2^I) \quad \dots \quad n(R_r^I)}{d_2 \quad \dots \quad d_r} = \chi \binom{n(R_1^I) \quad \dots \quad n(R_r^I)}{d_1 \quad \dots \quad d_r} \quad (3.25)$$

Using this and the induction hypothesis for $\beta_{R_{\geq 2}^I}(d - d_1, I_{\geq 2})$ we obtain the following equality:

$$\beta_R(d, I) = \sum_q q^{\sigma_R(I)-\chi} \binom{n(R_1^I) \quad \dots \quad n(R_r^I)}{d_1 \quad \dots \quad d_r} |J_{\mathbb{F}_q}|^{r-1} \prod_{k=1}^r \beta_{R_k^I}(d_k) \quad (3.26)$$

As

$$C(I; d_1, \dots, d_r) = \sigma_R(I) - \chi \binom{n(R_1^I) \quad \dots \quad n(R_r^I)}{d_1 \quad \dots \quad d_r} \quad (3.27)$$

the proof of the proposition is complete. \square

The recursive formula

The inductive expression for $\beta_R(d)$ can now be written as

$$f_R(q)\tau_{n(R)}(q) - \sum_{r \geq 2} \sum_{\{I | \ell(I)=r\}} \sum_{\circ} q^{C(I; d_1, \dots, d_r)} |J_{\mathbb{F}_q}|^{r-1} \prod_{k=1}^r \beta_{R_k^I}(d_k) \quad (3.28)$$

where

$$\tau_{n(R)}(q) = \frac{q^{(n(R)^2-1)(g-1)}}{q-1} Z_X(q^{-2}) \dots Z_X(q^{-n(R)}). \quad (3.29)$$

We now base change from \mathbb{F}_q to \mathbb{F}_{q^ν} . For the curve X_ν defined in the section 2, the $\beta_R(d, q^\nu)$ will be a function of q^ν and ω_i^ν for $i = 1, \dots, 2g$.

In the light of the induction formula and equation (2.17) we get that the function $\beta_R(d, q^\nu)$ is a polynomial in ω_i^ν for $i = 1, \dots, 2g$, and is a rational function in q^ν with the property that the denominator has factors only of the form $q^{\nu n_0}(q^{\nu n_1} - 1)(q^{\nu n_2} - 1) \dots (q^{\nu n_k} - 1)$, with $n_i \geq 1$ for $i \geq 1$. For such a functions, one can substitute $-t^{-1}$ for ω_i and t^{-2} for q , to obtain a new rational function. We denote this operation by $\phi \rightarrow \tilde{\phi}$. For example

$$\tilde{f}_R(t) = \frac{t^{-2 \dim \mathcal{F}_R} \prod_{i=1}^{n(R)} (1 - t^{2i})^{|S|}}{\prod_{P \in S} \prod_{\{i | R_i^P \neq 0\}} \prod_{l=1}^{R_i^P} (1 - t^{2l})} \quad (3.30)$$

and $\tilde{\tau}_{n(R)}(t)$ can be computed to be

$$\frac{t^{-2n(R)^2(g-1)} \prod_{i=1}^{n(R)} (1 + t^{2i-1})^{2g}}{(1 - t^{2n(R)}) \prod_{i=1}^{n(R)-1} (1 - t^{2i})^2} \quad (3.31)$$

This substitution is an important step in the computation of the Poincaré polynomial for the moduli space because of the proposition (4.34) of the next section.

Now we shall define rational functions $Q_{R,d}(t)$ and $Q_R(t)$ by

$$Q_{R,d}(t) = t^{n(R)^2(g-1)} (1 + t^{-1})^{2g} \tilde{\beta}_R(d) \quad (3.32)$$

and

$$Q_R(t) = t^{n(R)^2(g-1)} \tilde{f}_R(t) \tilde{\tau}_{n(R)}(t). \quad (3.33)$$

Observe that if the condition ‘par semi-stable = par stable’ is satisfied and if we define

$$P_{R,d}(t) = t^{2 \dim \mathcal{F}_R + 2(n(R)^2-1)(g-1)} (t^{-2} - 1) \tilde{\beta}_R(d) \quad (3.34)$$

then we have the relation

$$P_{R,d}(t) = \frac{t^{2 \dim \mathcal{F} + n(R)^2(g-1)}(1-t^2)}{(1+t)^{2g}} Q_{R,d}(t). \quad (3.35)$$

Now by proposition (4.34) and the fact that the dimension of the moduli space of parabolic semistable bundles is equal to $\dim \mathcal{F}_R + (n(R)^2 - 1)(g - 1)$, we get that $P_{R,d}$ is a power series in t which computes the Betti numbers of the moduli space of parabolic stable bundles with the given data R and degree d .

If we perform the tilde operation on the original formula, we get the following recursive formula.

Theorem 3.36. *The functions $Q_{R,d}$ and Q_R defined by (3.32) and (3.33) satisfy the following recursion formula.*

$$Q_R(t) = \sum_{r \geq 1} \sum_{\{I | \ell(I) = r\}} \sum_{\circ} t^{2N_R(I; d_1, \dots, d_r)} \prod_{k=1}^r Q_{R_k^I, d_k}(t) \quad (3.37)$$

where the second summation extends over all partitions I of R of length r , and where

$$N_R(I; d_1, \dots, d_r) = \sum_{k > l} (d_l n(R_k^I) - d_k n(R_l^I)) - \sum_{P \in S} \sum_{k > l, i > t} I_{i,k}^P I_{t,l}^P. \quad (3.38)$$

4 The substitution $\omega_i \rightarrow -t^{-1}$, $q \rightarrow t^{-2}$

In this section, we justify the substitution $\omega_i \rightarrow -t^{-1}$ and $q \rightarrow t^{-2}$, which gives us a recipe to compute the Poincaré polynomial of the moduli spaces, directly from the computation of the \mathbb{F}_q -rational points.

This substitution was briefly sketched in [H-N] for the rational function which counted the \mathbb{F}_q rational points of the moduli space of stable bundles when rank and degree are coprime. We formulate and prove this in a more general setup, which we have used in the body of the paper.

Let Y be a smooth projective variety over \mathbb{F}_q . Let $N_\nu = |Y(\mathbb{F}_{q^\nu})|$ and let $\omega_1, \dots, \omega_{2g}$ be fixed algebraic integers of norm $q^{1/2}$. Our basic assumption is

that N_ν is given by some formula

$$N_\nu = h(q^r, \omega_1^r, \dots, \omega_{2g}^r) \quad (4.1)$$

where $h(u, v_1, \dots, v_{2g})$ is a rational function of the form

$$\frac{p(u, v_1, \dots, v_{2g})}{u^{n_0}(u^{n_1} - 1) \dots (u^{n_k} - 1)} \quad (4.2)$$

where $p(u, v_1, \dots, v_{2g}) \in \mathbb{Z}[u, v_1, \dots, v_{2g}]$ is a polynomial with integral coefficients, and where $n_i \geq 1$ for all $i > 0$ and $n_0 \geq 0$. We wish to write down the Poincaré polynomial of Y in terms of the function h .

We first write down the function h as a suitable series and bound the coefficients. We can expand the numerator occurring in the expression for h as

$$p(u, v_1, \dots, v_{2g}) = \sum_{l=0}^N \sum_{|J|+2j=l} a_{J,j} v^J u^j \quad (4.3)$$

where J denotes the multi-index $J = (i_1, \dots, i_{2g})$, $|J| = \sum_{r=1}^{2g} i_r$, and $v^J = v_1^{i_1} \dots v_{2g}^{i_{2g}}$. Let $C > 0$ be any fixed integer such that $|a_{J,j}| < C$ for all J, j . The integer N in the summation above can be taken to be the ‘weighted degree’ of $p(u, v_1, \dots, v_{2g})$ where the variable u is given weight 2.

We can rewrite h as

$$\frac{1}{u^n (1 - u^{-n_1}) \dots (1 - u^{-n_k})} \sum_{l=0}^N \sum_{|J|+2j=l} a_{J,j} v^J u^j \quad (4.4)$$

where $n = \sum_{i=0}^k n_i$. Expanding each $1/(1 - u^{-n_i})$ as a power series in u^{-1} , we get

$$h = \frac{1}{u^n} \sum_{l=0}^N \sum_{|J|+2j'=l} \sum_{i \leq 0} a_{J,j'} b_i v^J u^{j'+i} \quad (4.5)$$

where b_i is the cardinality of the set of k -tuples of non-negative integers (a_1, \dots, a_k) such that $\sum_{r=1}^k a_r n(R) = -i$. Clearly, we have

$$b_i \leq (-i + 1)^k \quad (4.6)$$

Now the right hand side of the equation (4.5) becomes

$$\frac{1}{u^n} \sum_{l \leq N} \sum_{|J|+2j=l} \left(\sum_{j'+i=j} a_{J,j} b_i \right) v^J u^j. \quad (4.7)$$

Define

$$b_{J,j} = \sum_{j'+i=j} a_{J,j} b_i \quad (4.8)$$

This is a finite sum, which makes sense for every J, j such that $|J| + 2j \leq N$. In terms of these $b_{J,j}$, the expression for h can be written as

$$h = \frac{1}{u^n} \sum_{l \leq N} \sum_{|J|+2j=l} b_{J,j} v^J u^j. \quad (4.9)$$

Note that in the above series, there are only finitely many positive powers of u and infinitely many negative powers. The following lemma puts a bound on the coefficients $b_{J,j}$.

Lemma 4.10. *The coefficients $b_{J,j}$ as defined above satisfies the following inequality*

$$|b_{J,j}| \leq CN(N - j + 1)^k \quad (4.11)$$

Proof. One observes that

$$|b_{J,j}| \leq \sum_{j'+i=j} |a_{J,j} b_i| \leq C \sum_{j'+i=j} |b_i| \quad (4.12)$$

where j' and i are as in the preceding discussion. In the last expression of (4.12), the number of terms is $\leq N$, and by (4.6) each term $|b_i|$ is bounded by $(-i + 1)^k$. As $(-i + 1)^k \leq (N - j + 1)^k$ for $j' + i = j$, the last expression (4.12) is bounded by $CN(N - j + 1)^k$. This proves the lemma. \square

Let

$$\begin{aligned} h_{\geq 0}(u, v_1, \dots, v_{2g}) &= \sum_{l=2n}^N \sum_{|J|+2j=l} b_{J,j} v^J u^{j-n} \quad \text{if } N \geq 2n \\ &= 0 \quad \text{otherwise} \end{aligned} \quad (4.13)$$

and M_r be $h_{\geq 0}(q^r, \omega_1^r, \omega_2^r, \dots, \omega_{2g}^r)$, then these numbers are well defined because of lemma 1.

Let

$$Z_1(t) = \exp\left(\sum_{r \geq 1} M_r t^r / r\right) \quad (4.14)$$

and

$$Z_2(t) = \exp\left(\sum_{r \geq 1} (N_r - M_r) t^r / r\right), \quad (4.15)$$

then $Z_1(t)$ and $Z_2(t)$ are well defined formal power series. We also define

$$Z(t) = Z_1(t)Z_2(t). \quad (4.16)$$

Given any meromorphic function h on a disc in \mathbb{C} , let $\mu(h, \alpha)$ denote the number of zeros minus the number of poles h with norm α , counted with multiplicities.

Lemma 4.17. (a) $Z_2(t)$ is a non-vanishing holomorphic function on the disc $|t| < q^{1/2}$ and $Z_1(t)$ is a rational function, hence $Z(t)$ is a well defined meromorphic function in the region $|t| < q^{1/2}$, such that

$$\mu(Z(t), q^{-i/2}) = \mu(Z_1(t), q^{-i/2}). \quad (4.18)$$

(b) Let

$$P(T) = \sum_{i \geq 0} (-1)^{i+1} \mu(Z(t), q^{-i/2}) T^i, \quad (4.19)$$

then

$$P(T) = h_{\geq 0}(T^2, -T, -T, \dots, -T). \quad (4.20)$$

Proof. To prove that the function $Z_2(t)$ has the above mentioned property it is enough to verify that the function

$$g(t) := \sum_{r \geq 1} (N_r - M_r) t^r / r \quad (4.21)$$

is holomorphic on the disc $|t| < q^{1/2}$. This function is

$$\sum_{l < 2n} \sum_{|J| + 2j = l} b_{J,j} \omega^{Jr} q^{jr - nr} t^r / r. \quad (4.22)$$

The coefficient of t^r is equal to

$$\sum_{l < 2n} \sum_{|J|+2j=l} b_{J,j} \omega^{Jr} q^{jr-nr} / r \quad (4.23)$$

whose modulus is bounded by

$$\sum_{l < 2n} \sum_{|J|+2j=l} |b_{J,j}| q^{r(j-n+|J|/2)} / r \quad (4.24)$$

This by lemma (4.1) is

$$\leq \frac{NC}{rq^{nr}} \sum_{l < 2n} \sum_{|J|+2j=l} (N-j+1)^k q^{r(2j+|J|/2)} \quad (4.25)$$

$$\leq \frac{N^2C}{rq^{nr}} \sum_{l < 2n} q^{rl/2} ((3N+2-l)/2)^k \quad (4.26)$$

$$\leq \frac{N^2C}{2^k r} \sum_{l > 0} q^{-rl/2} (3N+2-2n+l)^k \quad (4.27)$$

which is clearly a finite sum for $r \geq 1$ because powers of q decay exponentially and the other term has polynomial growth. Now since $(a+l) \leq a^l$ for $a \geq 2$ therefore the above summation is bounded by

$$2^{-k} N^2 C \sum_{l > 0} ((3N+2-2n)^k / q^{r/2})^l / r. \quad (4.28)$$

Suppose r is large enough such that $q^{r/2} > 2(3N+2-2n)^k$ the coefficient of t^r has the bound $2^{-k+1} N^2 C (3N+2-2n)^k / (rq^{r/2})$ and the series with coefficient of t^r as above for large r clearly has radius of convergence $q^{r/2}$. Now we compute $Z_1(t)$ as

$$\exp \left(\sum_{r \geq 1} \sum_{l=2n}^N \sum_{|J|+2j=l} b_{J,j} \omega^{Jr} q^{jr-nr} t^r / r \right) \quad (4.29)$$

$$= \prod_{l=2n}^N \prod_{|J|+2j=l} \exp \left(b_{J,j} \sum_{r \geq 1} \omega^{Jr} q^{jr-nr} t^r / r \right) \quad (4.30)$$

which is equal to

$$\prod_{2n \leq l \leq N} \prod_{|J|+2j=l} (1 - \omega^J q^{j-n} t)^{(-1)b_{J,j}}, \quad (4.31)$$

hence this is a rational function, and this also proves that $Z(t)$ is a meromorphic function in the region $|t| < q^{1/2}$, and that $\mu(Z(t), q^{-i/2}) = \mu(Z_1(t), q^{-i/2})$. This finishes the proof of part (a).

Also from here we can read off that

$$\mu(Z_1(t), q^{-i/2}) = (-1) \sum_{|J|+2j+2n=i} b_{J,j}. \quad (4.32)$$

Clearly the polynomial $f_{\geq 0}(T^2, -T, -T, \dots, -T)$ now coincides with

$$\sum_{i \geq 0} (-1)^{i+1} \mu(Z_1(t), q^{-i/2}) T^i. \quad (4.33)$$

Now by part (a) the proof of the lemma is complete. \square

Now for the variety Y if the \mathbb{F}_q -rational points are given by the equation (4.1) and (4.2), we get that $Z(t)$ is the zeta function of Y and $P(t)$ is the Poincaré polynomial of Y . We can restate the lemma(4.17) in terms of the Poincaré polynomial of Y , using Poincaré duality, as follows.

Proposition 4.34. *The function $T^{2\dim(Y)} h(T^{-2}, -T^{-1}, -T^{-1}, \dots, -T^{-1})$ has a formal power series expansion $\sum_{\nu \geq 0} b_\nu T^\nu$ where b_ν is the ν^{th} -Betti number of Y for $\nu \leq 2\dim(Y)$.*

5 The Closed Formula

In this section we solve the recursion formula (theorem (3.36)) to obtain a closed formula for the Poincaré polynomial of the moduli space of parabolic stable bundles under the condition ‘par semi-stable = par stable’. We do this by generalizing the method of Zagier[Z] to the parabolic set up.

The induction formula can be re-written as

$$Q_R(x) = \sum_{r \geq 1} \sum_I \sum_{\circ} x^{n(R)(I; d_1, \dots, d_r)} \prod_{k=1}^r Q_{R_k^I, d_k}(x) \quad (5.1)$$

where $x = t^2$. The closed formula for $Q_{R,d}$ is given by the following theorem.

Theorem 5.2 *Let $Q_{R,d}$ and Q_R be formal Laurent series in $\mathbb{Q}((x))$ related by the formula (5.1). For any d and R we have*

$$Q_{R,d}(x) = \sum_{r \geq 1} \sum_I \frac{x^{M'_R(I; d) + M_R(I; (d + \alpha(R))/n(R))}}{(x^{n(R_1^I) + n(R_2^I)} - 1) \dots (x^{n(R_{r-1}^I) + n(R_r^I)} - 1)} \prod_{k=1}^r Q_{R_k^I}(x) \quad (5.3)$$

where $M'_R(I; d)$ and $M_R(I; \lambda)$ for a partition I of R and $\lambda \in \mathbb{R}$ are defined by

$$M'_R(I; d) = -(n(R) - n(R_r^I))d - \sigma_R(I) + (2n(R) - n(R_1^I) - n(R_r^I)) \quad (5.4)$$

$$\text{and } M_R(I; \lambda) = \sum_{k=1}^{r-1} (n(R_k^I) + n(R_{k+1}^I))[(n(R_1^I) + \dots + n(R_k^I))\lambda - \alpha(R_{\leq k}^I)]. \quad (5.5)$$

Here $[x]$ for a real number x denotes the largest integer less than or equal to x .

Proof. As in D. Zagier[Z] we introduce a real parameter with respect to which we perform a peculiar induction to prove the following theorem, which in turn implies theorem (5.2) by the substitution $\lambda = (d + \alpha(R))/n(R)$.

Theorem 5.6 *Let the hypothesis be as in the previous theorem. The two quantities*

$$Q_{R,d}^\lambda(x) = \sum_{r \geq 1} \sum_I \sum_{\circ_\lambda} x^{N_R(I; d_1, \dots, d_r)} \prod_{k=1}^r Q_{R_k^I, d_k}(x) \quad (5.7)$$

$$S_{R,d}^\lambda(x) = \sum_{r \geq 1} \sum_I \frac{x^{M'_R(I; d) + M_R(I; \lambda)}}{(x^{n(R_1^I) + n(R_2^I)} - 1) \dots (x^{n(R_{r-1}^I) + n(R_r^I)} - 1)} \prod_{k=1}^r Q_{R_k^I}(x) \quad (5.8)$$

agree for every real number $\lambda \geq (d + \alpha(R))/n(R)$.

Here \sum_{\circ_λ} denotes the summation over $(d_1, \dots, d_r) \in \mathbb{Z}^r$ such that $\sum_i d_i = d$ and the following holds

$$\lambda \geq \frac{d_1 + \alpha(R_1^I)}{n(R_1^I)} > \frac{d_2 + \alpha(R_2^I)}{n(R_2^I)} > \dots > \frac{d_r + \alpha(R_r^I)}{n(R_r^I)} \quad (5.9)$$

Proof. We first note that $Q_{R,d}^\lambda$ and $S_{R,d}^\lambda$ are step functions of λ and they only jump at a discrete subset of \mathbb{R} . We assume by induction that $Q_{R',d}^\lambda = S_{R',d}^\lambda$ for all data R' of rank $n(R') < n$, for all $d \in \mathbb{Z}$ and $\lambda \in \mathbb{R}$. Now for a given data R of rank $n(R) = n$ and $d \in \mathbb{Z}$ we make the following claims

Claim(1): given any N , there exists $\lambda_0(N)$ such that for $\lambda \geq \lambda_0(N)$, $Q_{R,d}^\lambda$ and $S_{R,d}^\lambda$ agree modulo x^N .

Claim(2): For any $\lambda \geq (d + \alpha(R))/n(R)$, if we define $Q_{R,d}^{\lambda-}$ (resp. $S_{R,d}^{\lambda-}$) to be $Q_{R,d}^{\lambda-\epsilon}$ (resp. $S_{R,d}^{\lambda-\epsilon}$) for $\epsilon > 0$ small enough such that the function $Q_{R,d}^\lambda$ (resp. $S_{R,d}^\lambda$) has no jumps in the interval $[\lambda - \epsilon, \lambda)$, then the two functions $\Delta Q_{R,d}^\lambda = Q_{R,d}^\lambda - Q_{R,d}^{\lambda-}$ and $\Delta S_{R,d}^\lambda = S_{R,d}^\lambda - S_{R,d}^{\lambda-}$ are equal.

Proof of claim(1): For a particular N , by equation (5.1), the coefficient of x^N in Q_R involves only finitely many choices of the integer r , partitions I , the integers (d_1, \dots, d_r) . Hence if we choose $\lambda_0(N) > (d_i + \alpha(R_i^I))/n_i$ for all such combinations of (r, I, d_1, \dots, d_r) , then the coefficient of x^N in Q_R and $Q_{R,d}^\lambda$ are equal for $\lambda \geq \lambda_0(N)$. On the other hand, if $r > 1$, we have $M_R(I; \lambda)$ occurring in the exponent of the numerator which tend to ∞ as λ tends to ∞ . So, for a fixed N if we choose λ large enough, we do not get any contribution for the coefficient of x^N in $S_{R,d}^\lambda$. But for $r = 1$ the part of the summation in $S_{R,d}^\lambda$ is just Q_R . hence the claim(1) follows.

Proof of claim(2):

Given a data R and a sub-data L of R we define a numerical function

$$\delta_R(L) = \sum_P \sum_{i>t} (R - L)_i^P L_t^P. \quad (5.10)$$

We first write down the recursions satisfied by the various numerical functions that we have encountered in the statement of the theorem (5.6).

Lemma 5.11 *Let I be a partition of R of length r . Let $0 < k < r$.*

$$\begin{aligned}
\text{(a)} \quad & \sigma_R(I) = \sigma_{R_{\leq k}^I}(I_{\leq k}) + \sigma_{R_{\geq k+1}^I}(I_{\geq k+1}) + \delta_R(R_{\leq k}^I) \\
\text{(b)} \quad & N_R(I; d_1, \dots, d_r) - N_{R_{\geq 2}^I}(I_{\geq 2}; d_2, \dots, d_r) \\
& = n(R_1^I)(n\lambda - d) - n\alpha(R_1^I) - \delta_R(R_1^I) \\
\text{(c)} \quad & M'_{R_{\leq k}^I}(I_{\leq k}; d(\lambda, R_{\leq k}^I)) + M'_{R_{\geq k+1}^I}(I_{\geq k+1}; d - d(\lambda, R_{\leq k}^I)) \\
& = M'_R(I; d) - (2n(R_{\leq k}^I) - n(R) - n(R_k^I) + n(R_r^I))d(\lambda, R_{\leq k}^I) \\
& \quad - n(R_k^I) - n(R_{k+1}^I) + \delta_R(R_{\leq k}^I) + n(R_{\leq k}^I)d.
\end{aligned} \tag{5.12}$$

where $d(\lambda, L) = n(L)\lambda - \alpha(L)$ for any data L .

Proof. All these statements follow from straight forward calculations, so we will not give the details. \square

We now compute $\Delta Q_{R,d}^\lambda$. It is zero unless there is a partition I of R and a r -tuple (d_1, \dots, d_r) with $\sum d_i = d$ such that $\lambda = (d_1 + \alpha(R_1^I))/n(R_1^I)$. For such a λ , we observe that

$$\Delta Q_{R,d}^\lambda = \sum_{r \geq 1} \sum_I \sum_{\substack{\circ \lambda \\ \lambda = (d_1 + \alpha(R_1^I))/n(R_1^I)}} x^{N_R(I; d_1, \dots, d_r)} \prod_{k=1}^r Q_{R_k^I, d_k}(x) \tag{5.13}$$

We can use the lemma (5.11) in the above formula and separate the expressions which have $k = 1$ and $k \geq 2$. Hence the right hand side in the equation (5.13) becomes

$$\sum_{\substack{L \text{ sub-data of } R \\ d(\lambda, L) \in \mathbb{Z}}} x^{n(L)(n(R)\lambda - d) - n(R)\alpha(L) - \delta_R(L)} Q_{L, d(\lambda, L)}^\lambda Q_{R-L, d-d(\lambda, L)}^{\lambda^-}. \tag{5.14}$$

Now we compute $\Delta S_{R,d}^\lambda$ at a λ when there is a jump. This happens when $(n(R_1^I) + \dots + n(R_k^I))\lambda - \alpha(R_k^I)$ is an integer for some partition I of R with $\ell(I) = r$ and for some positive integer $k < r$.

Fix a partition I of length r . Let

$$\pi_I = \{k < r \mid d(\lambda, R_{\leq k}^I) \in \mathbb{Z}\} \tag{5.15}$$

One can see that $\Delta M(I; \lambda) = \sum_{k \in \pi_I} (n(R_k^I) + n(R_{k+1}^I))$ so

$$\begin{aligned}
x^{M(I; \lambda)} - x^{M(I; \lambda^-)} &= x^{M(I; \lambda^-)} (x^{\sum_{k \in \pi_I} (n(R_k^I) + n(R_{k+1}^I))} - 1) \\
&= \sum_{k \in \pi_I} x^{M(I; \lambda^-) + \sum_{\{k' \in \pi_I | k' < k\}} (n(R_{k'}^I) + n(R_{k'+1}^I))} (x^{(n(R_k^I) + n(R_{k+1}^I))} - 1) \\
&= \sum_{k \in \pi_I} x^{M(R_{\leq k}^I; \lambda) + M(R_{\geq k+1}^I; \lambda^-) + (2n(R) - 2n(R_{\leq k}^I) + n(R_k^I) - n(R_r^I))d(\lambda, R_{\leq k}^I)} \\
&\quad \cdot (x^{(n(R_k^I) + n(R_{k+1}^I))} - 1)
\end{aligned} \tag{5.16}$$

Let $g_R(I; d)$ denote the following rational function of x

$$\frac{x^{M'_R(I; d) + M_R(I; \lambda)}}{(x^{n(R_1^I) + n(R_2^I)} - 1) \dots (x^{n(R_{r-1}^I) + n(R_r^I)} - 1)} \tag{5.17}$$

Using the lemma (5.11) and equation (5.16) we can verify that $\Delta(g_R(I; d))$ is equal to the following

$$\begin{aligned}
&\sum_{k \in \pi_I} x^{n(R_k^I)(n(R)\lambda - d) - n(R)\alpha(R_{\leq k}^I) - \delta_R(R_{\leq k}^I)} \\
&\quad \cdot g_{R_{\leq k}^I}(I_{\leq k}, d(\lambda, R_{\leq k}^I)) g_{R_{\geq k+1}^I}(I_{\geq k+1}, d - d(\lambda, R_{\leq k}^I))
\end{aligned} \tag{5.18}$$

Now $\Delta S_{R,d}^\lambda$ is computed to be

$$\sum_{r \geq 1} \sum_I \Delta(g_R(I; d)) \prod_{k=1}^r Q_{R_k^I}(x) \tag{5.19}$$

Using the equation (5.18), and grouping together all terms which give the sub-data L , we get the following expression for $\Delta S_{R,d}^\lambda$

$$\sum_L x^{n(L)(n(R)\lambda - d) - n(R)\alpha(L) - \delta_R(L)} S_{L,d_{\lambda,L}}^\lambda S_{R-L,d-d(\lambda,L)}^{\lambda^-} \tag{5.20}$$

where the summation is over sub-data L of R with $d(\lambda, L) \in \mathbb{Z}$. But $Q_{L,d_{\lambda,L}}^\lambda = S_{L,d_{\lambda,L}}^\lambda$ and $Q_{R-L,d-d(\lambda,L)}^{\lambda^-} = S_{R-L,d-d(\lambda,L)}^{\lambda^-}$ by induction (since $n(L)$ and $n(R) - n(L)$ are less than $n(R)$), hence we get $\Delta S_{R,d}^\lambda = \Delta Q_{R,d}^\lambda$. this proves claim(2).

To prove the theorem it is enough to check that the coefficient of x^N in $Q_{R,d}^\lambda$ and in $S_{R,d}^\lambda$ agree for any N . For a given N , the claim(1) implies that the

coefficients of $Q_{R,d}^\lambda$ and $S_{R,d}^\lambda$ are equal when λ is sufficiently large . Since $Q_{R,d}^\lambda$ and $S_{R,d}^\lambda$ are step functions of λ jumping only at a discrete set of real numbers, and for such real numbers by claim(2) their jumps agree therefore the jumps in the coefficients also agree, which in turn proves that the coefficients are the same. This completes the proof of the theorem. \square

Now if we define

$$\sigma'_R(I) = \sum_{P \in S} \sum_{k > l, i < t} I_{i,k}^P I_{t,l}^P \quad (5.21)$$

then one observes that dimensions of the flag varieties \mathcal{F}_R and $\mathcal{F}_{R_k^I}$ are related by

$$\dim \mathcal{F}_R - \sum_{k=1}^r \dim \mathcal{F}_{R_k^I} = \sigma_R(I) + \sigma'_R(I). \quad (5.22)$$

Using this expression we can formulate the closed formula for the Poincaré polynomial of the moduli space of parabolic semistable bundles as follows.

Theorem 5.23. *The Poincaré polynomial $P_{R,d}$ of the moduli space of parabolic stable bundles with a fixed determinant of degree d , and data R satisfying the condition ‘par semi-stable = par stable’ is given by*

$$\frac{1-t^2}{(1+t)^{2g}} \sum_{r \geq 1} \sum_I \frac{t^{2(\sigma'_R(I) - (n(R) - n(R_r^I))d + M_g(I; (d + \alpha(R))/n(R)))}}{(t^{2n(R_1^I) + 2n(R_2^I)} - 1) \dots (t^{2n(R_{r-1}^I) + 2n(R_r^I)} - 1)} \prod_{k=1}^r P_{R_k^I}(t) \quad (5.24)$$

where $M_g(I; \lambda)$ is

$$\sum_{k=1}^{r-1} (n(R_k^I) + n(R_{k+1}^I)) ((n(R_1^I) + \dots + n(R_k^I))\lambda - \alpha(R_{\leq k}^I)) + 1) + (g-1) \sum_{i < j} n(R_i^I) n(R_j^I) \quad (5.25)$$

and $P_R(t)$ is defined to be

$$\left(\frac{\prod_{i=1}^{n(R)} (1 - t^{2i})^{|S|}}{\prod_{P \in S} \prod_{\{i | R_i^P \neq 0\}} \prod_{l=1}^{R_i^P} (1 - t^{2l})} \right) \left(\frac{\prod_{i=1}^{n(R)} (1 + t^{2i-1})^{2g}}{(1 - t^{2n(R)}) \prod_{i=1}^{n(R)-1} (1 - t^{2i})^2} \right). \quad (5.26)$$

6 Sample calculations

Rank 2

Now we write down the Poincaré polynomial in more and more explicit forms for any data R such that $n(R) = 2$.

Let T be a subset of S defined by $\{P \in S | R_1^P = 1\}$, which is the set of vertices where the parabolic filtration is non-trivial. Then we get

$$P_R(t) = \frac{(1+t^2)^{|T|}(1+t)^{2g}(1+t^3)^{2g}}{(1-t^4)(1-t^2)} \quad (6.1)$$

Given any partition I of R , we define a subset T_I of T by

$$T_I = \{P \in T | I_{1,1}^P = 0\} \quad (6.2)$$

from this definition we observe that $\sigma'_R(I)$ (as defined in (5.21)) is just $|T_I|$.

Let $\chi_I : T \longrightarrow \{1, -1\}$ be defined by

$$\begin{aligned} \chi_I(P) &= 1 & \text{if } P \in T_I \\ &= -1 & \text{otherwise} \end{aligned} \quad (6.3)$$

Using the theorem (5.23) for rank 2 moduli we obtain the following.

Proposition 6.4. *For any degree d , the Poincaré polynomial for the moduli space of rank 2 parabolic bundles with data R and satisfying the condition ‘par semi-stable = par stable’ is given by*

$$P_{R,d}(t) = \frac{(1+t^2)^{|T|}(1+t^3)^{2g} - (\sum_I t^{2(g+|T_I|+[\psi_I]+a_I)})(1+t)^{2g}}{(1-t^4)(1-t^2)} \quad (6.5)$$

where

$$\psi_I = \sum_{P \in T} \chi_I(P)(\alpha_1^P - \alpha_2^P) \quad (6.6)$$

and a_I is 1 or 0 depending on whether $d + [\psi_I]$ is even or odd.

Now we put $g = 0$ in the formula. Since $P_{R,d}$ is a power series in t , one sees that $|T_I| + [\psi_I] + a_I \geq 0$ for every partition I

Using the proposition (6.4), the zeroth Betti number of the moduli space can be computed to be equal to $1 - |\{I | |T_I| + [\psi_I] + a_I = 0\}|$, hence the quantity

$|T_I| + [\psi_I] + a_I$ is 0 for at most one partition. Hence we obtain the following corollary

Corollary 6.7. *Assuming the condition ‘par semi-stable = par stable’ we have*

- a) *The moduli space of parabolic semistable bundles of rank 2 is non-empty iff for every partition I we have $|T_I| + [\psi_I] + a_I > 0$.*
- b) *The moduli is actually connected when it is non-empty.*

One can easily see that this condition is equivalent to the condition given by I.Biswas [B]. Even in higher rank we can get a criterion for existence of stable bundles by setting $P_{R,d}(t) \neq 0$.

In what follows we assume that ψ_I is never an integer, which has the effect that the condition ‘par semi-stable = par stable’ holds for all the degrees.

Using the above formula for the Poincaré polynomial we compute it in an explicit form, when the cardinality of S is small (1,2,3 and 4). For this, one observes that the above expression for $P_{R,d}(t)$, the dependence on the weights is only via their differences. In view of this we define $\delta^P = \alpha_1^P - \alpha_2^P$ for each $P \in S$.

When $S = \{P\}$, δ^P arbitrary, $R_i^P = 1$ for all i and any degree d , we compute the Poincaré polynomial to be

$$P_{R,d}(t) = \frac{(1+t^3)^{2g} - t^{2g}(1+t)^{2g}}{(1-t^2)^2}. \quad (6.8)$$

When $S = \{P_1, P_2\}$, δ^{P_1} and δ^{P_2} arbitrary, $R_i^P = 1$ for all i and P , and any degree d , we have

$$P_{R,d}(t) = \frac{(1+t^2)((1+t^3)^{2g} - t^{2g}(1+t)^{2g})}{(1-t^2)^2}. \quad (6.9)$$

When $S = \{P_1, P_2, P_3\}$, $R_i^{P_j} = 1$ for all i and $j = 1, \dots, 3$, and any degree d . By reordering P_1, P_2, P_3 , we may assume that $\delta^{P_1} \leq \delta^{P_2} \leq \delta^{P_3}$. Now there are two possibilities

- (i) If $\delta^{P_1} + \delta^{P_2} + \delta^{P_3} < -2$ or $-\delta^{P_1} + \delta^{P_2} + \delta^{P_3} > 0$ then

$$P_{R,d}(t) = \frac{(1+t^2)^2((1+t^3)^{2g} - t^{2g}(1+t)^{2g})}{(1-t^2)^2}. \quad (6.10)$$

(ii) If $\delta^{P_1} + \delta^{P_2} + \delta^{P_3} > -2$ and $-\delta^{P_1} + \delta^{P_2} + \delta^{P_3} < 0$ (which is the remaining case), then

$$P_{R,d}(t) = \frac{(1+t^2)^2(1+t^3)^{2g} - 4t^{2g+2}(1+t)^{2g}}{(1-t^2)^2}. \quad (6.11)$$

When $S = \{P_1, P_2, P_3, P_4\}$, $R_i^{P_j} = 1$ for all i and j , and any degree d . Again we assume $\delta^{P_1} \leq \delta^{P_2} \leq \delta_3^P \leq \delta^{P_4}$. Again there are two possibilities

(i) If $\delta^{P_1} + \delta^{P_2} + \delta^{P_3} - \delta^{P_4} < -2$ or $-\delta^{P_1} + \delta^{P_2} + \delta^{P_3} + \delta^{P_4} > 0$ then

$$P_{R,d}(t) = \frac{(1+t^2)^3((1+t^3)^{2g} - t^{2g}(1+t)^{2g})}{(1-t^2)^2}. \quad (6.12)$$

(ii) If $\delta^{P_1} + \delta^{P_2} + \delta^{P_3} - \delta^{P_4} > -2$ and $-\delta^{P_1} + \delta^{P_2} + \delta^{P_3} + \delta^{P_4} < 0$ (which is the remaining case), then

$$P_{R,d}(t) = \frac{(1+t^2)^3(1+t^3)^{2g} - 4t^{2g+2}(1+t^2)(1+t)^{2g}}{(1-t^2)^2}. \quad (6.13)$$

Remark 6.14. Note that in these cases we have considered, the Poincaré polynomial does not depend on the degree. In fact we can verify that in general for rank 2 the Poincaré polynomial is independent of the degree.

Rank 3 and 4

Using the software Mathematica, we have computed the Poincaré polynomials and the Betti numbers for the rank 3 and rank 4 when the number of parabolic points is one or two. In this case we find that the Poincaré polynomial has dependence on the weights and degree. In the appendix we actually give the tables for the Betti numbers (in the rank 3 and rank 4 case) taking different set of weights into consideration. For the Poincaré polynomial we choose one set of weights as an example in each of the following cases.

When rank = 3, $S = \{P_1\}$, $R_i^{P_1} = 1$ for all i and assume that the condition ‘par semi-stable = par stable’ holds. Then for all choices of weights and degree d we have

$$P_{R,d} = \{t^{6g-2}(1+t^2+t^4)(1+t)^{4g} - t^{4g-2}(1+t^2)^2(1+t)^{2g}(1+t^3)^{2g} + (1+t^3)^{2g}(1+t^5)^{2g}\} / ((t^2-1)^4(1+t^2)). \quad (6.15)$$

When $\text{rank} = 3$, $S = \{P_1, P_2\}$, $R_i^{P_j} = 1$ for all i and j . One observes that the condition ‘par semi-stable = par stable’ holds for all choices of degree. For $(\alpha_1^{P_1}, \alpha_2^{P_1}, \alpha_3^{P_1}) = (0, 1/12, 3/12)$, $(\alpha_1^{P_2}, \alpha_2^{P_2}, \alpha_3^{P_2}) = (1/12, 5/12, 6/12)$, one observes that the condition ‘par semi-stable = par stable’ holds for all choices of degree. When the degree $d = 0$ or $2 \bmod 3$ we find that

$$P_{R,d} = \{-3t^{4g}(1+t)^{2g}(1+t^2)^2(1+t^3)^{2g} + t^{6g}(1+t)^{4g}(2+5t^2+2t^4) + (1+t^3)^{2g}(1+t^5)^{2g}(1+t^2+t^4)\} / (1-t^2)^4 \quad (6.16)$$

and if $d = 1 \bmod 3$ then the Poincaré polynomial is

$$P_{R,d} = (1+t^2+t^4) \{t^{6g-2}(1+t^2+t^4)(1+t)^{4g} - t^{4g-2}(1+t^2)^2(1+t)^{2g}(1+t^3)^{2g} + (1+t^3)^{2g}(1+t^5)^{2g}\} / (1-t^2)^4. \quad (6.17)$$

When rank is 4 and $|S| = 1$ we find that the Poincaré Polynomial depends on the degree too. If we choose $R_i^P = 1$ for all i , and choose $(\alpha_1^P, \alpha_2^P, \alpha_3^P, \alpha_4^P) = (0, 1/8, 1/4, 1/2)$, then the condition ‘par semi-stable = par stable’ holds for all choices of degree. We find that that

$$P_{R,0} = P_{R,1} = P_{R,2} = \{(1+t^3)^{2g}(1+t^5)^{2g}(1+t^7)^{2g} - 2t^{-2+6g}(1+t)^{2g}(1+t^3)^{2g}(1+t^5)^{2g}(1+t^2+t^4) - t^{-4+8g}(1+t)^{2g}(1+t^3)^{4g}(1+t^2+t^4)^2 + t^{-4+10g}(1+t^2)(1+t)^{4g}(1+t^3)^{2g}(3+5t^2+5t^4+3t^6) - 2t^{-4+12g}(1+t)^{6g}(1+t^2+t^4)^2\} / ((1-t^2)^6(1+t^2)(1+t^2+t^4)) \quad (6.18)$$

and

$$P_{R,3} = \{(1+t^3)^{2g}(1+t^5)^{2g}(1+t^7)^{2g} - t^{-4+6g}(1+t)^{2g}(1+t^3)^{2g}(1+t^5)^{2g}(1+t^2+t^4)(1+t^4) - t^{-4+8g}(1+t)^{2g}(1+t^3)^{4g}(1+t^2+t^4)^2 + t^{-6+10g}(1+t^2)^4(1+t)^{4g}(1+t^3)^{2g}(1+t^4) - 2t^{-6+12g}(1+t)^{6g}(1+t^4)(1+t^2+t^4)^2\} / ((1-t^2)^6(1+t^2)(1+t^2+t^4)). \quad (6.19)$$

If we choose the weights $(\alpha_1^P, \alpha_2^P, \alpha_3^P, \alpha_4^P) = (0, 1/5, 4/5, 9/10)$ then again the condition ‘par semi-stable = par stable’ holds for all choices of degree. If $P_{R,d}$ and $P'_{R,d}$ denote the Poincaré polynomial for the moduli space of parabolic stable bundles with data R (satisfying $n(R)=4$), having degree d and with weights $(0, 1/8, 1/4, 1/2)$ and $(0, 1/5, 4/5, 9/10)$ respectively, then we find that $P'_{R,0} = P'_{R,2} = P_{R,0}$ and $P'_{R,1} = P'_{R,3} = P_{R,1}$.

7 Appendix : Betti number tables

The following tables give the Betti numbers up to the middle dimension of the moduli space of parabolic bundles over X for rank 2, 3 and 4 and low genus. When $\beta_0 = 0$, we mean that the space is empty.

Rank 2

Any degree d , $R_i^P = 1$ for all i and $P \in S$.

Case A) $S = \{P_1\}$, δ^{P_1} arbitrary.

Case B) $S = \{P_1, P_2\}$, δ^{P_1} and δ^{P_2} arbitrary.

Case C) $S = \{P_1, P_2, P_3\}$, $\delta^{P_1} + \delta^{P_2} + \delta^{P_3} < -2$ or $-\delta^{P_1} + \delta^{P_2} + \delta^{P_3} > 0$

Case D) $S = \{P_1, P_2, P_3\}$, $\delta^{P_1} + \delta^{P_2} + \delta^{P_3} > -2$ and $-\delta^{P_1} + \delta^{P_2} + \delta^{P_3} < 0$.

Case E) $S = \{P_1, P_2, P_3, P_4\}$, $\delta^{P_1} + \delta^{P_2} + \delta^{P_3} - \delta^{P_4} < -2$ or $-\delta^{P_1} + \delta^{P_2} + \delta^{P_3} + \delta^{P_4} > 0$

Case F) $S = \{P_1, P_2, P_3, P_4\}$, $\delta^{P_1} + \delta^{P_2} + \delta^{P_3} - \delta^{P_4} > -2$ and $-\delta^{P_1} + \delta^{P_2} + \delta^{P_3} + \delta^{P_4} < 0$.

	Genus g=0						Genus g=1					
	A	B	C	D	E	F	A	B	C	D	E	F
β_0	0	0	0	1	0	1	1	1	1	1	1	1
β_1	-	-	-	-	-	0	0	0	0	0	0	0
β_2	-	-	-	-	-	-	-	2	3	4	4	5
β_3	-	-	-	-	-	-	-	-	0	2	0	2
β_4	-	-	-	-	-	-	-	-	-	-	6	8

	Genus g=2						Genus g=3					
	A	B	C	D	E	F	A	B	C	D	E	F
β_0	1	1	1	1	1	1	1	1	1	1	1	1
β_1	0	0	0	0	0	0	0	0	0	0	0	0
β_2	2	3	4	4	5	5	2	3	4	4	5	5
β_3	4	4	4	4	4	4	6	6	6	6	6	6
β_4	2	4	7	8	11	12	3	5	8	8	12	12
β_5	-	8	12	16	16	20	12	18	24	24	30	30
β_6	-	-	8	14	15	22	18	21	26	27	34	35
β_7	-	-	-	-	24	32	12	24	42	48	66	72
β_8	-	-	-	-	-	-	-	36	57	72	83	99
β_9	-	-	-	-	-	-	-	-	48	68	90	116
β_{10}	-	-	-	-	-	-	-	-	-	-	114	144

Rank 3

Case A) $S = \{P\}$, $R_i^P = 1$ for all i .

We take all choices of weights and degrees.

Case B) When $S = \{P_1, P_2\}$, $R_i^{P_1} = 1 = R_i^{P_2}$ for all i .

$(\alpha_1^{P_1}, \alpha_2^{P_1}, \alpha_3^{P_1}) = (0, 1/12, 3/12)$, $(\alpha_1^{P_2}, \alpha_2^{P_2}, \alpha_3^{P_2}) = (1/12, 5/12, 6/12)$

$d=0$ or $2 \pmod 3$

Case C) $S = \{P_1, P_2\}$, $R_i^{P_1} = 1 = R_i^{P_2}$ for all i .

$(\alpha_1^{P_1}, \alpha_2^{P_1}, \alpha_3^{P_1}) = (0, 1/12, 3/12)$, $(\alpha_1^{P_2}, \alpha_2^{P_2}, \alpha_3^{P_2}) = (1/12, 5/12, 6/12)$,

$d=1 \pmod 3$.

	A, g =				B, g =				C, g =			
	0	1	2	3	0	1	2	3	0	1	2	3
β_0	0	1	1	1	0	1	1	1	0	1	1	1
β_1	-	0	0	0	-	0	0	0	-	0	0	0
β_2	-	2	3	3	-	5	5	5	-	4	5	5
β_3	-	0	4	6	-	2	4	6	-	0	4	6
β_4	-	-	7	7	-	12	15	15	-	8	15	15
β_5	-	-	16	24	-	6	24	36	-	0	24	36
β_6	-	-	18	28	-	16	40	49	-	10	39	49
β_7	-	-	36	60	-	-	80	120	-	-	76	120
β_8	-	-	45	103	-	-	108	176	-	-	98	176
β_9	-	-	56	140	-	-	188	314	-	-	164	314
β_{10}	-	-	70	261	-	-	251	531	-	-	203	530
β_{11}	-	-	64	354	-	-	344	784	-	-	264	778
β_{12}	-	-	-	537	-	-	436	1312	-	-	318	1293
β_{13}	-	-	-	780	-	-	480	1878	-	-	332	1828
β_{14}	-	-	-	998	-	-	528	2816	-	-	370	2697
β_{15}	-	-	-	1380	-	-	-	4036	-	-	-	3788
β_{16}	-	-	-	1652	-	-	-	5454	-	-	-	4983
β_{17}	-	-	-	1936	-	-	-	7442	-	-	-	6610
β_{18}	-	-	-	2170	-	-	-	9346	-	-	-	8007
β_{19}	-	-	-	2160	-	-	-	11526	-	-	-	9572
β_{20}	-	-	-	-	-	-	-	13394	-	-	-	10812
β_{21}	-	-	-	-	-	-	-	14562	-	-	-	11508
β_{22}	-	-	-	-	-	-	-	15210	-	-	-	11984

Rank 4

$$|S| = 1$$

Case A) $R_i^P = 1$ for all i , $d = 0$ or 1 or $2 \pmod{4}$,

$$(\alpha_1^P, \alpha_2^P, \alpha_3^P, \alpha_4^P) = (0, 1/8, 1/4, 1/2) \quad \text{Or}$$

$R_i^P = 1$ for all i , $d = 0$ or $2 \pmod{4}$,

$$(\alpha_1^P, \alpha_2^P, \alpha_3^P, \alpha_4^P) = (0, 1/5, 4/5, 9/10).$$

Case B) $R_i^P = 1$ for all i , $d = 3 \pmod{4}$,

$$(\alpha_1^P, \alpha_2^P, \alpha_3^P, \alpha_4^P) = (0, 1/8, 1/4, 1/2) \quad \text{Or}$$

$R_i^P = 1$ for all i , $d = 1$ or $3 \pmod{4}$,

$$(\alpha_1^P, \alpha_2^P, \alpha_3^P, \alpha_4^P) = (0, 1/5, 4/5, 9/10).$$

	A,g			B,g		
	0	1	2	0	1	2
β_0	0	1	1	0	1	1
β_1	-	0	0	-	0	0
β_2	-	4	4	-	3	4
β_3	-	2	4	-	0	4
β_4	-	8	11	-	5	11
β_5	-	4	20	-	0	20
β_6	-	10	31	-	6	31
β_7	-	-	64	-	-	64
β_8	-	-	90	-	-	89
β_9	-	-	164	-	-	160
β_{10}	-	-	241	-	-	232
β_{11}	-	-	376	-	-	356
β_{12}	-	-	563	-	-	521
β_{13}	-	-	792	-	-	712
β_{14}	-	-	1144	-	-	1001
β_{15}	-	-	1508	-	-	1272
β_{16}	-	-	2003	-	-	1635
β_{17}	-	-	2492	-	-	1952
β_{18}	-	-	2989	-	-	2263
β_{19}	-	-	3424	-	-	2528
β_{20}	-	-	3675	-	-	2660
β_{21}	-	-	3816	-	-	2760
β_{22}	-	-	-	-	-	-

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